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On the Existence and Approximation of Invariant Densities for Nonsingular Transformations on $\mathbb{R}^d$

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Let T be a piecewise monotone, expanding, and C^2 mapping of the unit interval to itself which admits an absolutely continuous invariant measure $\nu = f\,dm$. S. Ulam has described a sequence of finite dimensional operators P_n approximating the Frobenius–Perron operator associated to T, and conjectured that the sequence of non-negative fixed points f_n obtained for the P_n converge strongly to f. This was shown to be the case by T. Y. Li. A. Boyarsky and S. Y. Lou gave a partial generalization of this result to the case of expanding, C^2 Jablonski transformations on the multidimensional unit cube, obtaining weak approximation of the invariant density. In this article we replace weak with strong convergence in the multidimensional result using a compactness criterion due to Kolmogorov. We also discuss both existence and approximation of the invariant density in the case of general nonsingular transformations on $\mathbb{R}^d$ using the approximating sequence of Ulam.

I. INTRODUCTION

Let $(X, \mathcal{F}, \mu)$ be a Lebesgue probability space and let $T\colon X \to X$ be a measurable, nonsingular mapping. (This means $T^{-1}A \in \mathcal{F}$ whenever $A \in \mathcal{F}$ and $\mu \circ T^{-1} \ll \mu$.) In this setting the *invariant measure problem* (IMP) asks: Does there exist a T-invariant measure $\nu \ll \mu$?

Associated to T are two positive operators $S_T\colon L^\infty(X) \to L^\infty(X)$ and $P_T\colon L^1(X) \to L^1(X)$ defined by $S_T g = g \circ T$ for all $g \in L^\infty$ and $\int_A P_T f\,d\mu = \int_{T^{-1}A} f\,d\mu$ for all $A \in \mathcal{F}$ and $f \in L^1$. P_T is called the *Frobenius–Perron operator* associated to T. The IMP is equivalent to the existence of an $f \in L^1(X)$, $f \geq 0$ and $\|f\|_1 = 1$ with $P_T f = f$ (an invariant density). We remark that $P_T^* = S_T$ so $\|P_T\| = 1$ and $1 \in \text{Spectrum}(P_T)$ although 1 need not be an eigenvalue admitting a non-negative eigenvector. (See, for example, Adler [1] for an early example and Gora and Schmitt [5] for a recent, more delicate example of transformations T which do not admit

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finite, absolutely continuous invariant measures.) On the other hand, there is an extensive literature describing additional conditions on T which ensure an affirmative answer to the IMP.

In case $(X, \mathcal{F}, \mu) = ([0,1], \mathcal{F}, m)$, the unit interval with Borel subsets $\mathcal{F}$ and Lebesgue measure m, S. Ulam [8] observed that P_T is the continuous analogue of a natural action P_n on the n-dimensional subspace of $L^1([0,1])$ consisting of step functions formed with respect to a uniform partition of $[0,1]$ into n intervals. Each P_n may be shown to have a non-negative fixed point f_n and he asked if, provided T is known to admit an invariant density f, does the (normalized) sequence f_n converge to f in L^1?

T. Y. Li [7] gave a positive answer to Ulam's question for the class of piecewise C^2 and expanding transformations on the unit interval. In addition he suggested that the fixed points f_n may give a more efficient approximation scheme for the invariant density f than that afforded by the orbit of a single point and application of the ergodic theorem. His paper contains some numerical evidence supporting this view for the transformation

$$T(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ \frac{7}{4} - \frac{3}{2}x, & \frac{1}{2} \le x \le 1. \end{cases}$$

In [2], A. Boyarski and S. Y. Lou investigated Ulam's question for the class of C^2 Jablonski transformations (defined in Jablonski [6]) on the d-dimensional unit cube. Unfortunately, their methods led only to weak approximation of the invariant density. The purpose of this note is to present a full generalization of Li's strong approximation result to the class of C^2 Jablonski transformations. We also observe that existence of an invariant density is not necessary for the approximation result to hold. This is because the principal issue for such an approximation, namely, precompactness of the sequence of finite dimensional fixed points f_n, is already sufficient to solve the IMP, although as we shall see, this is not particularly useful in the specific case of C^2 Jablonski transformations, providing only another solution of the IMP by way of the variational inequality presented in [6].

We close this section by stating in a general setting the existence and approximation result on which the remainder of this article is based.

THEOREM 1.1. *Let T be a nonsingular transformation on $(X, \mathcal{F}, \mu)$. Assume that there is a sequence of bounded linear operators P_n on $L^1(X)$ satisfying*:

(1) *$P_n \to P_T$ in strong operator topology*

(2) *For each n there exists $f_n \ge 0$, $\|f_n\|_1 = 1$ so $P_n f_n = f_n$.*

Then any limit point f of the sequence $\{f_n\}_{n=1}^{\infty}$ is the density of an absolutely continuous T-invariant probability measure on X.

COROLLARY 1.2. *If the sequence $\{f_n\}$ is precompact in $L^1(X)$ then T admits an absolutely continuous, invariant probability measure $f\,d\mu$ and there is a subsequence $f_{n_k} \to f$.*

COROLLARY 1.3. *If the sequence $\{f_n\}$ is precompact in $L^1(X)$ and T has a* unique, *absolutely continuous invariant probability measure $f\,d\mu$, then $f_n \to f$.*

PROOF OF THEOREM 1.1. Let us assume, by dropping to a subsequence, that $f_n \to f$. That $f \geq 0$ a.e. and that $\|f\|_1 = 1$ are immediate. By the uniform boundedness principle, $\sup_n \|P_n\| < \infty$, so the fact that $P_T f = f$ follows from the estimate.

$$\|P_T f - f\|_1 \leq \|P_T f - P_n f\|_1 + \|P_n f - P_n f_m\|_1$$
$$+ \|P_n f_m - P_m f_m\|_1 + \|f_m - f\|. \quad \blacksquare$$

II. THE INVARIANT MEASURE PROBLEM: EXISTENCE AND APPROXIMATION

For the remainder of this article we shall restrict our discussion to the case $X = I^d = [0,1]^d$, the d-dimensional unit cube, $\mathscr{F}$ the Borel subsets on I^d, and m_d the d-dimensional Lebesgue measure. We begin by describing Ulam's approximating sequence of finite rank operators. For the most part, we shall adopt the notation of [2]. L^1 will denote the space $L^1(I^d, m_d)$ and we will write $\|\cdot\|$ for $\|\cdot\|_1$.

Let $I^d = \bigcup_{k=1}^{l^d} I_k$ where each I_k is a cube of the form $I_k = \prod_{i=1}^{d}[r_i/l, (r_i+1)/l)$, $0 \leq r_i < l$. For $1 \leq s,\ t \leq l^d$ set $p_{st} = m_d(I_s \cap T^{-1}I_t)/m_d(I_s)$. Let Δ_l be the l^d-dimensional subspace of L^1 generated by the characteristic functions $\{\chi_{I_k}\}_{k=1}^{l^d}$. (We identify Δ_l in the natural way with $\mathbb{R}^{l^d}$.) Let $f \in \Delta_l$, say $f = \sum_s \alpha_s \chi I_s$ and let $P_l\colon \Delta_l \to \Delta_l$ be defined by the formula

$$P_l f(x) = \sum_t \chi_{I_t}(x) \sum_s \alpha_s p_{st}.$$

That is, P_l acts as right multiplication of the vector $f = (\alpha_1, \ldots, \alpha_{l^d}) \in \Delta_l$ by the matrix $(p_{st})_{1 \le s,t \le l^d}$.

Let Q_l denote the conditional expectation operator from L^1 to Δ_l,

$$Q_l f(x) = \sum_{s=1}^{l^d} \chi_{I_s}(x) \frac{1}{m_d(I_s)} \int_{I_s} f \, dm_d.$$

The connection between P_l and P_T is contained in the following, whose proof is a straightforward calculation to be found in [2].

Lemma 2.1. *For each* $f \in \Delta_l$, $P_l f = Q_l P_T f$.

Remark 2.2. The essential point is that P_l is constructed using a uniform partition. The same formula would be true for any P_l constructed with respect to a partition of I^d into atoms of equal measure.

Since the matrix (p_{st}) has non-negative entries and row sums 1, the classical Frobenius–Perron theory (see Gantmacher [4] for example) yields

Proposition 2.3. *For each* $l \ge 1$ *there exists* $f \in \Delta_l$, $f \ne 0$, $f \ge 0$, *and* $P_l f = f$. *Without loss of generality we may assume* $\|f\| = 1$.

We extend P_l to an operator on L^1 in the natural way

$$\bar{P}_l f = P_l Q_l f.$$

Proposition 2.4. *For each* $f \in L^1$, $\bar{P}_l f \xrightarrow{l \to \infty} P_T f$.

Proof. Enumerate the sequence of rectangular partitions by $I^d = \bigcup_{k=1}^{l^d} I_k^{(l)}$, $l = 1, 2, \ldots$. Recall that $\|Q_l\| = 1$. Also, for all s, $\mathrm{diam}(I_s^{(l)}) = \sqrt{d}/l \xrightarrow{l \to \infty} 0$, hence $Q_l f \xrightarrow{l \to \infty} f$ strongly for each $f \in L^1$. Now observe

$$\begin{aligned}\|\bar{P}_l f - P_T f\| &= \|Q_l P_T Q_l f - P_T f\| \\ &\le \|Q_l P_T Q_l f - Q_l P_T f\| + \|Q_l P_T f - P_T f\| \\ &\le \|Q_l f - f\| + \|Q_l P_T f - P_T f\|\end{aligned}$$

with both terms tending to zero as $l \to \infty$. ∎

We have shown that Ulam's approximating sequence $\bar{P}_l$ satisfies the conditions of Theorem 1.1. Indeed, the setting can be more general as the only restrictions needed in this section have been that the P_l be constructured with respect to finite partitions $\mathscr{P}_l$ with atoms of uniform measure and diameters tending to zero. If the partitions $\mathscr{P}_l$ form a sequence of refinements we may weaken the latter condition to $\bigvee_{l=1}^{\infty}\mathscr{P}_l = \mathscr{F}$.

III. The Case of C^2 Jablonski Transformations

We continue to adopt the notation of [2]. Let $\beta = \{D_1, D_2, \ldots, D_p\}$ be a finite partition of I^d into disjoint rectangles of the form $D_j = \prod_{i=1}^{d}[a_{ij}, b_{ij})$. The transformation T is assumed to satisfy

(I) (Independence) The ith coordinate of the point $T(x_1, \ldots, x_d)$ depends only on the rectangular atom D_j containing $(x_1, \ldots, x_d)$ and the value of x_i. More precisely, if $(x_1, \ldots, x_d) \in D_j$ then

$$T(x_1, \ldots, x_d) = \left(\phi_{1j}(x_1), \ldots, \phi_{dj}(x_d)\right),$$

where each ϕ_{ij}: $[a_{ij}, b_{ij}) \to [0, 1]$ for $1 \leq i \leq d$ and $1 \leq j \leq p$.

(S) (Smoothness) The ϕ_{ij} are C^2 functions on their domains.

(E) (Expansiveness)

$$\lambda = \inf_{ij}\left\{\inf_{x \in [a_{ij}, b_{ij})} \left|\phi'_{ij}(x)\right|\right\} > 1.$$

It follows that T maps each D_j bijectively onto a d-dimensional subrectangle in I^d with $m_d(TD_j) \geq \lambda^d m_d(D_j)$. These transformations are in some sense the simplest multidimensional generalization of the piecewise monotone and expanding interval maps. They were defined and investigated in Jablonski [6] and are known in the literature as C^2 *Jablonski Transformations*. The goal in this section is to prove the following.

Theorem 3.1. *Let T: $I^d \to I^d$ be a C^2 Jablonski transformation with $\lambda > 2$ and let $\overline{P}_l$ be the approximating sequence of finite rank operators described in the preceding section. For each l, denote by f_l a fixed point for P_l with $f_l \geq 0$ and $\|f_l\| = 1$. Then there exists an $f \in L^1$ and subsequence $f_{l_k} \to f$ so $P_T f = f$. If T admits a unique absolutely continuous invariant probability measure $\nu = f\,dm_d$ then $f_l \to f$.*

We begin by reviewing a few known results about C^2 Jablonski transformations.

Let $A = \prod_{i=1}^{d}[a_i b_i]$ and let $g\colon A \to \mathbb{R}$. Set, for each i, $1 \le i \le d$

$$\bigvee_i^A g(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$$

$$= \sup\left\{ \sum_{k=1}^{r} \left| g\left(x_1, x_2, \ldots, x_{i-1}, x_i^k, x_{i+1}, \ldots, x_d\right) - g\left(x_1, x_2, \ldots, x_{i-1}, x_i^{k-1}, x_{i+1}, \ldots, x_d\right)\right| \text{ where } a_i = x_i^0 < x_i^1 < \cdots < x_i^r = b_i \right\}.$$

Define $\pi_i\colon \mathbb{R}^d \to \mathbb{R}^{d-1}$, $\pi_i(x_1, \ldots, x_d) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$. If $f\colon A \to \mathbb{R}$ set

$$\mathop{\mathrm{W}}_i^A f = \inf\left\{ \int_{\pi_i(A)} \bigvee_i^A g\, dm_{d-1} \,\middle|\, f = g m_d \text{ a.e. and } \bigvee_i^A g \text{ measurable} \right\}.$$

Finally, set $\mathop{\mathrm{W}}^A f = \max_{1 \le i \le d} \mathop{\mathrm{W}}_i^A f$. If $\mathop{\mathrm{W}}^A f < \infty$ we say that f_n is of bounded *Tonelli variation* on A. Let $\mathscr{E}$ be the set of functions of the form

$$g = \sum_{j=1}^{m} g_j \chi_{A_j},$$

where each $A_j \subseteq I^d$ is a d-dimensional rectangle and $g_j\colon I^d \to \mathbb{R}$ are C^1 functions on A_j.

The crucial variational inequality is to be found in the following.

THEOREM 3.1a. (Jablonski [6]). *Let T satisfy* (I), (S), *and* (E) *above, and assume $\lambda > 2$. Then there exist constants K_T and $\alpha = 2/\lambda < 1$ so that for all $f \in \mathscr{E}$*

$$\mathop{\mathrm{W}}^{I^d} P_T f \le K_T \|f\| + \alpha \mathop{\mathrm{W}}^{I^d} f.$$

We will also need to know that application of the conditional expectation operator does not increase the Tonelli variation.

LEMMA 3.2 [2, Lemma 6]. *If $f \in L^1$ then $\mathop{\mathrm{W}}^{I^d} Q_l f \le \mathop{\mathrm{W}}^{I^d} f$.*

This lemma, combined with the previous theorem yields

LEMMA 3.3 [2, Lemma 7]. *Let T be as in Theorem* 3.1. *For each n, let f_n be a fixed point of the approximating finite dimensional operator P_n given by Lemma* 2.1. *Then $\{\mathop{\mathrm{W}}^{I^d} f_n\}_{n=1}^{\infty}$ is bounded.*

We sketch the proof for completeness.

$$\bigvee^{I^d} f_n = \bigvee^{I^d} Q_n P_T f_n \leq \bigvee^{I^d} P_T f_n \leq K_T \|f_n\| + \alpha \bigvee^{I^d} f_n.$$

Hence $\bigvee^{I^d} f_n \leq K_T/(1 - \alpha) < \infty$. ∎

Remark 3.4. The proof in [2] of Lemma 3.2 is a more or less straightforward induction on the case $d = 1$ which was already presented in [7]. Unfortunately, the one dimensional proof offered in [7] is incorrect. However, an equally elementary and correct proof may be easily supplied by the reader.

Precompactness of the collection of fixed points will be a consequence of the following result due to Kolmogorov (see [3, IV.8.21] for its proof). Theorem 3.1 will then follow immediately from Corollaries 1.2 and 1.3 and this discussion in Section 2.

THEOREM 3.5. *Let $\mathfrak{R} \subseteq L^1(\mathbb{R}^n)$ be a norm bounded set of functions and assume the following limits are attained uniformly over $f \in \mathfrak{R}$:*

(1) $$\lim_{\substack{\Delta x \to 0 \\ \Delta x \in \mathbb{R}^n}} \int_{\mathbb{R}^n} |f(x + \Delta x) - f(x)| \, dx = 0$$

(2) $$\lim_{N \to \infty} \int_{\mathbb{R}^n - \{[-N, N]^n\}} |f(x)| \, dx = 0.$$

Then $\mathfrak{R}$ is a precompact subset of $L^1(\mathbb{R}^n)$.

If $f \colon I^d \to \mathbb{R}$ define $\bar{f} \colon \mathbb{R}^d \to \mathbb{R}$ by

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in I^d \\ 0 & \text{if } x \in \mathbb{R}^d - I^d. \end{cases}$$

If $\mathfrak{R} = \{\bar{f}_l\}_{l=1}^{\infty}$, where $\bar{f}_l$ are extensions of the fixed points f_l, then condition (2) in the above theorem is trivially satisfied. Condition (1) will first be examined in the case $d = 1$. In the following $V_0^1 f$ denotes the usual variation of $f \colon [0, 1] \to \mathbb{R}$.

PROPOSITION 3.6. *Let $f \in L^1([0, 1], m)$. Then for $0 < |\Delta x| < 1$*

$$\int_{\mathbb{R}} |\bar{f}(x + \Delta x) - \bar{f}(x)| \, dx \leq 2|\Delta x| \{V_0^1 f + \|f\|_1\}.$$

2 is the best possible constant in the above inequality.

Proof. Without loss of generality we assume $V_0^1 f < \infty$. Let us also assume first that f is non-decreasing and $\Delta x > 0$. Then

$$\begin{aligned}\int_{\mathbb{R}} |\bar{f}(x+\Delta x) - \bar{f}(x)| &= \int_{-\Delta x}^{0} |\bar{f}(x+\Delta x)|\,dx + \int_0^{1-\Delta x} \bar{f}(x+\Delta x) - \bar{f}(x)\,dx \\ &\quad + \int_{1-\Delta x}^{1} |\bar{f}(x)|\,dx \\ &= \int_0^{\Delta x} |f(x)| - f(x)\,dx + \int_{1-\Delta x}^{1} |f(x)| + f(x)\,dx.\end{aligned}$$

If f does not change sign on $[0,1]$ we may upperbound the last expression by

$$2\Delta x \|f\|_\infty \le 2\Delta x\{V_0^1 f + |f(t_0)|\},$$

where $t_0 \in [0,1]$ in an arbitrary point. On the other hand, if $f(0) < 0$ and $f(1) > 0$ then we have the same expression upperbounded by

$$2\,\Delta x|f(0)| + 2\,\Delta x|f(1)| + 2\,\Delta x\{V_0^1 f\}.$$

Combining these, with similar calculations for $\Delta x < 0$ one has, for non-decreasing, bounded f and $0 < |\Delta x| < 1$

$$\int_{\mathbb{R}} |\bar{f}(x+\Delta x) - \bar{f}(x)|\,dx \le 2|\Delta x|\{V_0^1 f + |f(t_0)|\},$$

where $t_0 \in [0,1]$ is arbitrary.

If $V_0^1 f < \infty$ pick $t_0 \in [0,1]$ so $|f(t_0)| \le \|f\|_1$ and write $f = g - h$, g, h non-decreasing, bounded, and $h(t_0) = 0$. Since $\bar{f} = \bar{g} - \bar{h}$ we obtain, using the above estimates,

$$\begin{aligned}\int_{\mathbb{R}} |\bar{f}(x+\Delta x) - \bar{f}(x)|\,dx &\le 2|\Delta x|\{V_0^1 g + V_0^1 h + |g(t_0)| + |h(t_0)|\} \\ &\le 2|\Delta x|\{V_0^1 f + |f(t_0)|\} \\ &\le 2|\Delta x|\{V_0^1 f + \|f\|_1\}. \quad \blacksquare\end{aligned}$$

Before proceeding with the multidimensional version of this lemma we introduce some additional notation. If $\Delta x \in \mathbb{R}^d$, say $\Delta x = (\Delta x_1, \Delta x_2, \ldots, \Delta x_d)$ and if $x \in \mathbb{R}^d$ we set $x^0 = x$ and for $1 \le i \le d$, $x^i = (x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_i + \Delta x_i, x_{i+1}, \ldots, x_d)$.

LEMMA 3.7. *Let $f \in L^1$. Then for $\Delta x \in \mathbb{R}^d$, $\|\Delta x\|_1 < 1$, and $\Delta x \neq 0$*

$$\int_{\mathbb{R}^d} |\bar{f}(x + \Delta x) - \bar{f}(x)|\, dm_d(x) \leq 2\sqrt{d}\,\|\Delta x\|_1 \left\{ \overset{I^d}{\mathbb{W}} f + \|f\| \right\}.$$

Proof. Again, we assume without loss of generality $\overset{I^d}{\mathbb{W}} f < \infty$ and observe

$$\int_{\mathbb{R}^d} |\bar{f}(x + \Delta x) - \bar{f}(x)|\, dm_d(x) \leq \sum_{i=1}^{d} \int_{\mathbb{R}^d} |\bar{f}(x^i) - \bar{f}(x^{i-1})|\, dm_d(x)$$

$$= \sum_{i=1}^{d} \int_{\mathbb{R}^d} |\bar{f}_i(x^i) - \bar{f}_i(x^{i-1})|\, dm_d(x), \tag{*}$$

where each f_i has been chosen so $f = f_i\ m_d$ – a.e., $\overset{I^d}{\underset{i}{\bigvee}} f_i$ is measurable, and

$$\int_{[0,1]^{d-1}} \overset{I^d}{\underset{i}{\bigvee}} f_i\, dm_{d-1} \leq \underset{i}{\mathbb{W}} f + \varepsilon$$

$$\leq \mathbb{W} f + \varepsilon.$$

Each of the summands in (*) is estimated as follows. Since

$$\int_{\mathbb{R}^d} |\bar{f}_i(x^i) - \bar{f}_i(x^{i-1})|\, dm_d(x)$$

$$= \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} |\bar{f}_i(x^i) - \bar{f}_i(x^{i-1})|\, dm(x_i)\, dm_{k-1} \tag{**}$$

and

$$|\bar{f}_i(x^i) - \bar{f}_i(x^{i-1})|$$

$$= |\bar{f}_i(x_1 + \Delta x_i, \ldots, x_i + \Delta x_i, x_{i+1}, \ldots, x_d)$$

$$- \bar{f}_i(x_1 + \Delta x_i, \ldots, x_{i-1} + \Delta x_{i-1}, x_i, \ldots, x_d)|,$$

Lemma 3.6 gives the following upperbound on terms (**):

$$\int_{\mathbb{R}^{d-1}} 2|\Delta x_i| \left\{ \bigvee_i^{I^d} f_i(x_1 + \Delta x_1, \dots, x_{i-1} + \Delta x_{i-1}, x_i, \dots, x_d) + \int_{\mathbb{R}} |\bar{f}_i(x_1 + \Delta x_1, \dots, x_{i-1} + \Delta x_{i-1}, x_i, \dots, x_d)| \, dm(x_i) \right\} dm_{d-1}$$

$$\le 2|\Delta x_i| \left\{ \bigvee\!\!\!\bigvee_i^{I^d} f + \varepsilon \right\} + 2|\Delta x_i| \int_{I^d} |f_i(x_1, \dots, x_d)| \, dm_d$$

$$\le 2|\Delta x_i| \left\{ \bigvee\!\!\!\bigvee^{I^d} f + \|f\| + \varepsilon \right\}.$$

Summing these estimates on each term in (*) and observing that $\varepsilon > 0$ was arbitrary yields the desired inequality. ∎

This lemma completes the proof of Theorem 3.1. There is a partial generalization of the theorem to the case $1 < \lambda \le 2$, which we now describe. For such λ pick k so $\lambda^k > 2$ and let $\phi = T^k$. It is well known that an invariant density for ϕ may be used to construct an invariant density for T—we omit a discussion of this and turn our attention to its approximation. Let $P_n(\phi)$ be the finite rank operators approximating P_ϕ and (by dropping to a subsequence) assume that f_n are fixed points for $P_n(\phi)$ with $f_n \to f$. Let $P_n(T)$ be the approximating sequence for P_T, and observe that

$$(P_n(T))^l f_n \overset{n \to \infty}{\to} P_T^l f$$

for each $0 \le l < k$. Conclude that

$$h_n = \frac{1}{k} \left\{ f_n + P_n(T) f_n + \cdots + (P_n(T))^{k-1} f_n \right\}$$

satisfies $h_n \in \Delta_n$ and $h_n \to g = (1/k)\{f + \cdots + P_T^{k-1} f\}$. Since $h_n \ge 0$, $h_n \ne 0$ we may set

$$\bar{f}_n = \frac{1}{\|h_n\|} h_n$$

so that

$$\bar{f}_n \in \Delta_n, \bar{f}_n \to \frac{1}{\|g\|} g = \bar{f}, \bar{f} \ge 0, \|\bar{f}\| = 1, \text{ and } P_T \bar{f} = \bar{f}.$$

References

1. R. L. Adler, *F*-expansions revisited, *in* "Lecture Notes in Mathematics," Vol. 318, pp. 1–5, Springer-Verlag, New York/Berlin, 1973.
2. A. Boyarsky and S. Y. Lou, Approximating measures invariant under higher-dimensional chaotic transformations, *J. Approx. Theorem* **65** (1991), 231–244.
3. N. Dunford and J. Schwartz, "Linear Operators, Part I," Interscience, NY, 1963.
4. F. R. Gantmacher, "Application of the Theory of Matrices," Interscience, NY, 1959.
5. P. Gora and B. Schmitt, Un example de transformation dilatante et C^1 par morceaux de l'intervalle, sans probabilité absolument continue invariante, *Ergodic Theory Dynamical Systems* **9** (1989), 101–113.
6. M. Jablonski, On invariant measures for piecewise C^2-transformations of the n-dimensional cube, *Ann. Polon. Math.* **43** (1983), 185–195.
7. T. Y. Li, Finite approximation for Frobenius-Perror operator: A solution to Ulam's conjecture, *J. Approx. Theory* **17** (1976), 177–186.
8. S. Ulam, "Problems in Mathematics," Interscience, NY, 1960.